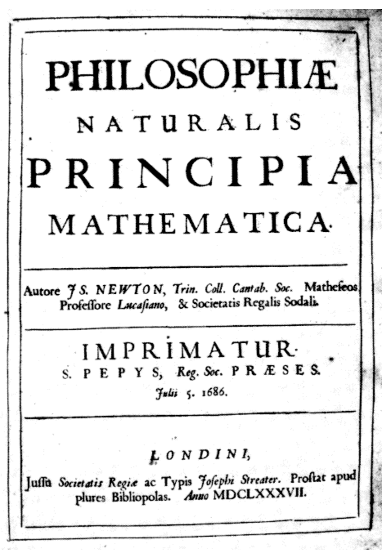
Newton's Laws of Motion



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore J. S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, Reg. Soc. PRÆSES.
Julii 5. 1686.

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Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

"The Mathematical Principles of Natural Philosophy" 1687

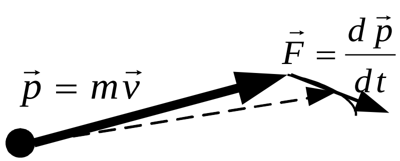
<u>First Law</u>

Everybody continues in its state of rest or of uniform motion in straight line unless it is compelled to change that state by forces impressed upon it.

$$\vec{F} = 0 \quad \rightarrow \quad \vec{v} = \overrightarrow{const}$$

<u>Second Law</u>

The rate of change of momentum is proportional to the force impressed and in the same direction as that force.



$$\vec{F} = \frac{d}{dt}\vec{p} = \frac{d}{dt}(m\vec{v})$$

<u>Third Law</u>

To every action there is always opposed an equal reaction.

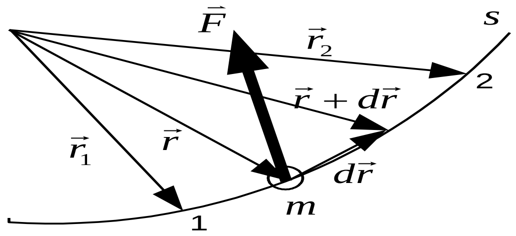
$$\vec{F}_{12} = -\vec{F}_{21}$$

Work and Energy

The work W of a force $\vec{F}$ acting on a particle m that moves as a result of this along a curve s from $\vec{r}_1$ to $\vec{r}_2$ is defined by:

$$W_{12} \overset{\Delta}{=} \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{d}{dt}\left(m\dot{\vec{r}}\right) \cdot d\vec{r}$$

$d\vec{r}$ ***is the <u>displacement</u> on a real path.***



The kinetic energy T is defined as:

$$T \overset{\Delta}{=} \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}}$$

For a constant mass m

$$W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{d}{dt}\left(m\dot{\vec{r}}\right) \cdot d\vec{r} = m\int_{\vec{r}_1}^{\vec{r}_2} \left(\frac{d}{dt}\dot{\vec{r}}\right) \cdot \dot{\vec{r}}\, dt = \frac{m}{2}\int_{\dot{\vec{r}}_1}^{\dot{\vec{r}}_2} d\left(\dot{\vec{r}} \cdot \dot{\vec{r}}\right) = T_2 - T_1$$

Work and Energy (continue)

When the force depends on the position alone, i.e. $\vec{F} = \vec{F}(\vec{r})$, ***and the quantity*** $\vec{F} \cdot d\vec{r}$ ***is a perfect differential***

$$\boxed{\vec{F}(\vec{r}) \cdot d\vec{r} = -dV(\vec{r})}$$

The force field is said to be <u>conservative</u> and the function $V(\vec{r})$ ***is known as the <u>Potential Energy</u>. In this case:***

$$\boxed{W_{12} \stackrel{\Delta}{=} \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = -\int_{\vec{r}_1}^{\vec{r}_2} dV(\vec{r}) = V(\vec{r}_1) - V(\vec{r}_2) = V_1 - V_2}$$

The work W_{12} ***does not depend on the path from*** $\vec{r}_1$ ***to*** $\vec{r}_2$ ***. It is clear that in a conservative field, the integral of*** $\vec{F} \cdot d\vec{r}$ ***over a closed path is zero.***

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \underbrace{\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}}_{path1} + \underbrace{\int_{\vec{r}_2}^{\vec{r}_1} \vec{F} \cdot d\vec{r}}_{path2} = (V_1 - V_2) + (V_2 - V_1) = 0}$$

Using Stoke's Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{s}$ ***it means that***

$$\boxed{rot\ \vec{F} = \nabla \times \vec{F} = 0}$$

Therefore $\vec{F}$ ***is the gradient of some scalar function*** $\vec{F} = -\nabla V(\vec{r})$

$$\boxed{\vec{F} \cdot d\vec{r} = -dV = -\nabla V(\vec{r}) \cdot d\vec{r}}$$

Work and Energy (continue)

and

$$\frac{dV}{dt} = \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = -\lim_{\Delta t \to 0} \vec{F} \cdot \frac{d\vec{r}}{dt} = -\vec{F} \cdot \dot{\vec{r}}$$

But also for a constant mass system

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}}\right) = \frac{m}{2}\left(\dot{\vec{r}} \cdot \ddot{\vec{r}} + \ddot{\vec{r}} \cdot \dot{\vec{r}}\right) = m \ddot{\vec{r}} \cdot \dot{\vec{r}} = \vec{F} \cdot \dot{\vec{r}}$$

Therefore for a <u>constant mass in a conservative field</u>

$$\frac{d}{dt}(T + V) = 0 \quad \Rightarrow \quad T + V = Total\ Energy = const.$$

The Principal Laws of Analytical Dynamics

The basic laws of dynamics can be formulated (expressed mathematically) in several ways other that that given by Newton's Laws. The most important are:

(a) D'Alembert Principle

(b) Lagrange's Equations

(c) Hamilton's Equations

(d) Hamilton's Principle

All are basically equivalent.

Basic Definitions

Given a system of N particles defined by their coordinates:

$$\vec{r}_k = \vec{r}_k(x_k, y_k, z_k) = x_k(t)\vec{i} + y_k(t)\vec{j} + z_k(t)\vec{k} \quad k = 1,2,\cdots,N$$

where $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors defining any Inertial Coordinate System

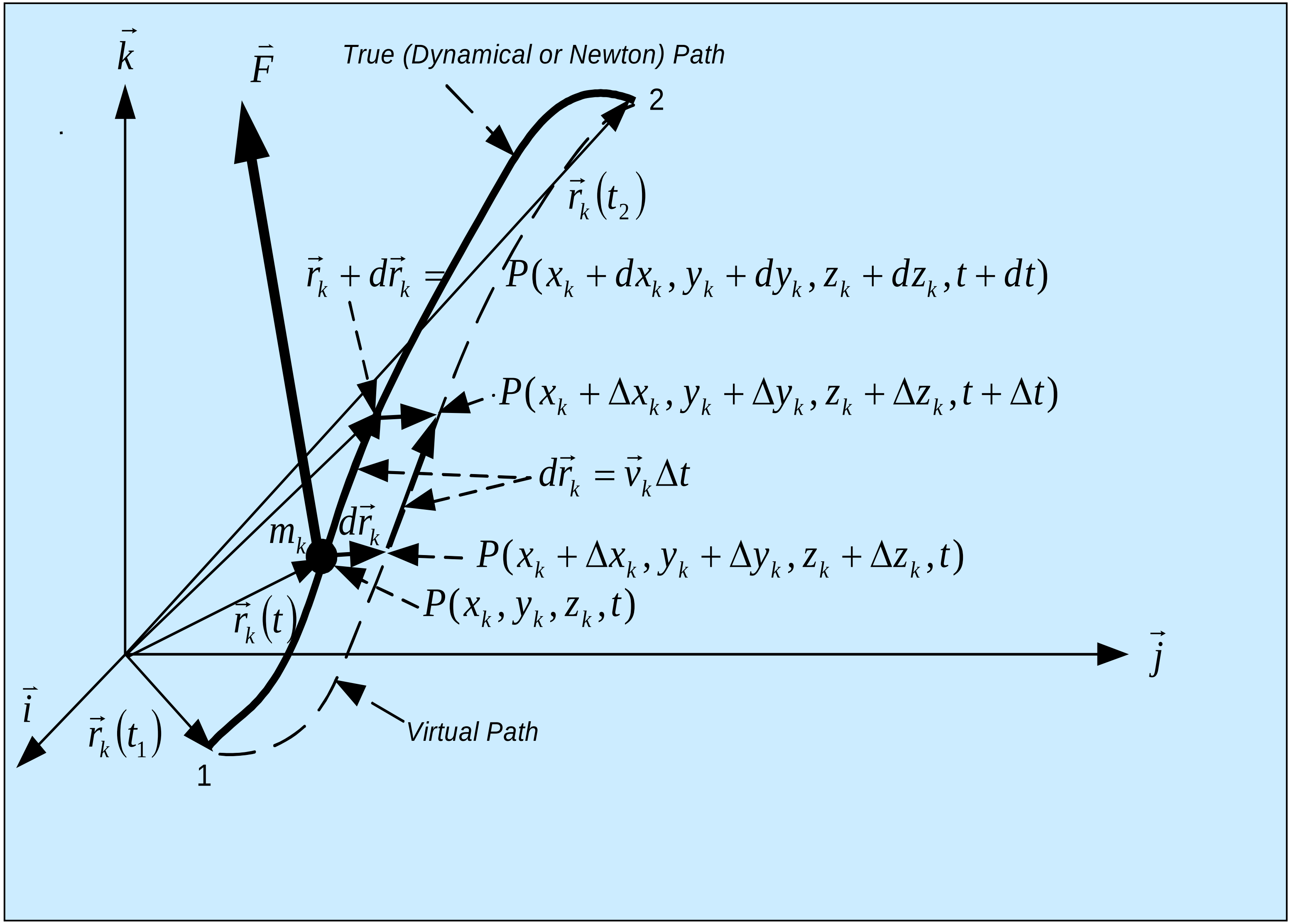
The real displacement of the particle m_k :

$$d\vec{r}_k = dx_k(t)\vec{i} + dy_k(t)\vec{j} + dz_k(t)\vec{k} \quad k = 1,2,\cdots,N$$

is the infinitesimal change in the coordinates along real path caused by all the forces acting on the particle m_k .

The virtual displacements $(\Delta x_k, \Delta y_k, \Delta z_k, \Delta t)$ are infinitesimal changes in the coordinates; they are not real changes because they are not caused by real forces. The virtual displacements define a virtual path that coincides with the real one at the end points.

Basic Definitions (continue)



Constraints

If the N particles $\vec{r}_k(x_k, y_k, z_k) \quad k = 1,2,\cdots,N$ *are free the system has* ***n = 3 N degrees of freedom***.

The constraints on the system can be of the following types:

(1) *Equality Constraints*

(2) *Inequality Constraints*

(1) *Equality Constraints: The general form (the Pffafian form)*

$$\sum_{k=1}^{N} \left[a_{xk}^{l}(\vec{r},t)\,dx_k + a_{yk}^{l}(\vec{r},t)\,dy_k + a_{zk}^{l}(\vec{r},t)\,dz_k \right] + a_t^{l}(\vec{r},t)\,dt = 0 \qquad l = 1,2,\cdots,m$$

or

$$\sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot d\vec{r}_k + a_t^{l} dt = 0 \qquad l = 1,2,\cdots,m \qquad rank\left\{a_{xk}^{l}, a_{yk}^{l}, a_{zk}^{l}\right\} = m$$

We can classify the constraints as follows:

(*a*) *Time Dependency*

(a1) *Catastatic* $\quad a_t^{l} = 0 \qquad l = 1,2,\cdots,m$

(a2) *Acatastatic* $\quad a_t^{l} \neq 0 \qquad l = 1,2,\cdots,m$

Constraints (continue)

Equality Constraints: The general form (the Pffafian form) (continue)

$$\sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot d\vec{r}_k + a_t^l dt = 0 \qquad l = 1,2,\cdots,m \qquad rank\left\{a_{xk}^l, a_{yk}^l, a_{zk}^l\right\} = m$$

(*b*) *Integrability*

(b1) *<u>Holonomic</u> if the <u>Pffafian forms</u> are integrable; i.e.:*

$$df_l = \sum_{k=1}^{N}\left(\frac{\partial f_l}{\partial x_k} d\,x_k + \frac{\partial f_l}{\partial y_k} d\,y_k + \frac{\partial f_l}{\partial z_k} d\,z_k\right) + \frac{\partial f_l}{\partial t} dt \qquad l = 1,2,\cdots,m$$

or

$$f_l\left(x_1, y_1, z_1, \cdots, x_N, y_N, z_N, t\right) = 0 \qquad l = 1,2,\cdots,m$$

(b2) *<u>Non-holonomic</u> if the <u>Pffafian forms</u> are not integrable*

(b2.1) *Scleronomic*:

$$\left.\frac{\partial f_l}{\partial t} = 0\right|_{l=1,2,\cdots,m}$$

or

$$f_l\left(x_1, y_1, z_1, \cdots, x_N, y_N, z_N\right) = 0 \qquad l = 1,2,\cdots,m$$

(b2.2) *Rheonomic*:

$$\left.\frac{\partial f_l}{\partial t} \neq 0\right|_{l=1,2,\cdots,m}$$

Constraints (continue)

(2) *Inequality Constraints*:

(*a*) *Stationary Boundaries* (*time independent*):

$$f_l(x_1, y_1, z_1, \cdots, x_N, y_N, z_N) \geq 0 \qquad l = 1,2,\cdots,m$$

(*b*) *Non-stationary Boundaries* (*time dependent*):

$$f_l(x_1, y_1, z_1, \cdots, x_N, y_N, z_N, t) \geq 0 \qquad l = 1,2,\cdots,m$$

Constraints (continue)

Displacements Consistent with the Constraints:

The real displacement $d\vec{r}_k = dx_k\,\vec{i} + dy_k\,\vec{j} + dz_k\,\vec{k}, dt$ ***consistent with the General Equality Constraints (Pffafian form) is:***

$$\sum_{k=1}^{N}\left[a^l_{xk}\,dx_k + a^l_{yk}\,dy_k + a^l_{zk}\,dz_k\right] + a^l_t\,dt = \sum_{k=1}^{N}\vec{a}^{\,l}_k \cdot d\vec{r}_k + a^l_t\,dt = 0 \quad l = 1,2,\cdots,m$$

The virtual displacement $\Delta\vec{r}_k = \Delta x_k\,\vec{i} + \Delta y_k\,\vec{j} + \Delta z_k\,\vec{k}, \Delta t$ ***consistent with the General Equality Constraints (Pffafian form) is:***

$$\sum_{k=1}^{N}\left[a^l_{xk}\,\Delta x_k + a^l_{yk}\,\Delta y_k + a^l_{zk}\,\Delta z_k\right] + a^l_t\,\Delta t = \sum_{k=1}^{N}\vec{a}^{\,l}_k \cdot \Delta\vec{r}_k + a^l_t\,\Delta t = 0 \quad l = 1,2,\cdots,m$$

Dividing the Pffafian equation by dt and taking the limit, we obtain:

$$a^l_t = -\sum_{k=1}^{N}\vec{a}^{\,l}_k \cdot \dot{\vec{r}}_k \quad l = 1,2,\cdots,m$$

Now replace a^l_t ***in the virtual displacement equation***

$$\sum_{k=1}^{N}\vec{a}^{\,l}_k \cdot \left(\Delta\vec{r}_k - \dot{\vec{r}}_k\,\Delta t\right) = 0 \quad l = 1,2,\cdots,m$$

Define the δ variation as:

$$\delta \overset{\Delta}{=} \Delta - \Delta t\,\frac{d}{dt}$$

Constraints (continue)

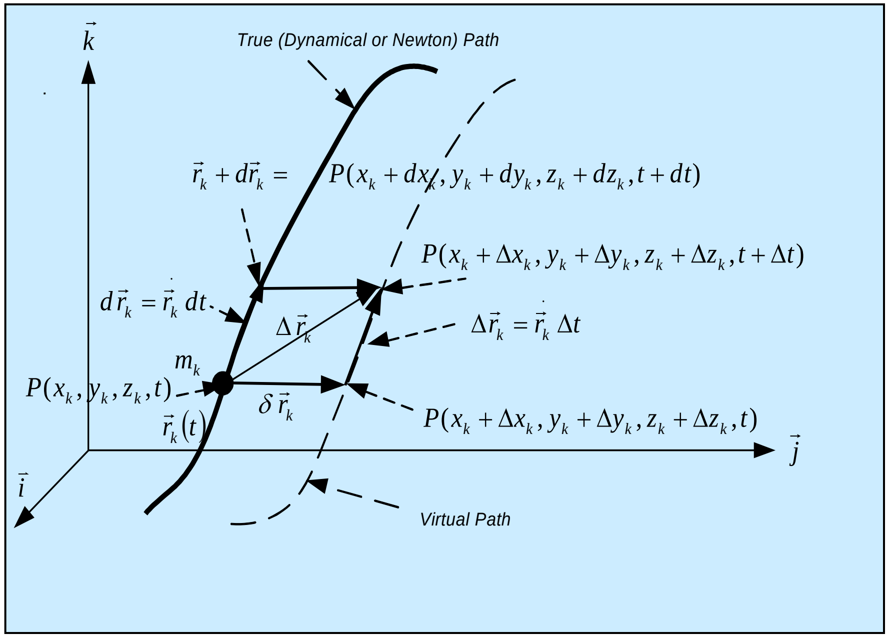
Displacements Consistent with the Constraints (continue):

Define the δ variation as:

$$\delta \overset{\Delta}{=} \Delta - \Delta t \frac{d}{d\,t}$$

Then:

$$\delta \vec{r}_k \overset{\Delta}{=} \Delta \vec{r}_k - \Delta t \frac{d}{d\,t} \vec{r}_k$$



From the Figure we can see that δ variation corresponds to a virtual displacement in which the time t is held fixed and the coordinates varied to the constraints imposed on the system.

$$\sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \delta \vec{r}_k = 0 \quad l = 1,2,\cdots,m$$

For the Holonomic Constraints:

$$f_l\left(x_1, y_1, z_1, \cdots, x_N, y_N, z_N, t\right) = 0 \qquad l = 1,2,\cdots,m$$

$$\sum_{k=1}^{N} \nabla_k f_l \cdot \delta \vec{r}_k = 0 \quad l = 1,2,\cdots,m$$

Generalized Coordinates

The motion of a mechanical system of N particles is completely defined by n = 3N coordinates $x_k(t), y_k(t), z_k(t) \quad (k = 1,2,\cdots,N)$. Quite frequently we may find it more advantageous to express the motion of the system in terms of a different set of coordinates, say $\vec{q} = (q_1, q_2, \cdots, q_n)^T$. If we take in consideration the m constraints we can reduce the coordinates to n = 3N-m generalized coordinates.

$$\vec{r}_k(\vec{q},t) = \vec{r}_k(q_1,q_2,\cdots,q_n,t) = x_k(\vec{q},t)\vec{i} + y_k(\vec{q},t)\vec{j} + z_k(\vec{q},t)\vec{k} \quad k = 1,2,\cdots,N$$

$$d\vec{r}_k = \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j} dq_j + \frac{\partial \vec{r}_k}{\partial t} dt = dx_k \vec{i} + dy_k \vec{j} + dz_k \vec{k} \quad k = 1,2,\cdots,N$$

$$\vec{v}_k = \dot{\vec{r}}_k = \frac{d\vec{r}_k}{dt} = \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_k}{\partial t} \quad k = 1,2,\cdots,N$$

In the same way

$$\Delta\vec{r}_k = \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j} \Delta q_j + \frac{\partial \vec{r}_k}{\partial t} \Delta t = \Delta x_k \vec{i} + \Delta y_k \vec{j} + \Delta z_k \vec{k} \quad k = 1,2,\cdots,N$$

and

$$\delta\vec{r}_k = \Delta\vec{r}_k - \dot{\vec{r}}_k \Delta t = \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j} \Delta q_j + \frac{\partial \vec{r}_k}{\partial t} \Delta t - \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j} \dot{q}_j \Delta t - \frac{\partial \vec{r}_k}{\partial t} \Delta t = \quad k = 1,2,\cdots,N$$

Generalized Coordinates (continue)

$$\delta \vec{r}_k = \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j}\left(\Delta q_j - \dot{q}_j \Delta t\right) = \sum_{j=1}^{n} \frac{\partial \vec{r}_k}{\partial q_j}\delta q_j \qquad k = 1,2,\cdots,N$$

where $\delta q_j \overset{\Delta}{=} \Delta q_j - \dot{q}_j \Delta t$

The Generalized Equality Constraints in Generalized Coordinates will be:

$$\sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot d\vec{r}_k + a_t^l dt = \sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i} dq_i + \left(a_t^l + \sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \frac{\partial \vec{r}_k}{\partial t} \right) dt =$$
$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \frac{\partial \vec{r}_k}{\partial q_i} \right) dq_i + \left(a_t^l + \sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \frac{\partial \vec{r}_k}{\partial t} \right) dt = 0 \qquad l = 1,2,\cdots,m$$

If we define

$$c_i^l \overset{\Delta}{=} \sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \sum_{i=1}^{n} \frac{\partial \vec{r}_k}{\partial q_i} \qquad \& \qquad c_t^l \overset{\Delta}{=} a_t^l + \sum_{k=1}^{N} \vec{a}_k^{\,l} \cdot \frac{\partial \vec{r}_k}{\partial t}$$

we obtain $\sum_{i=1}^{n} c_i^l \, dq_i + c_t^l dt = 0 \qquad l = 1,2,\cdots,m$

and the virtual displacements compatible with the constraints are

$$\sum_{i=1}^{n} c_i^l \, \delta q_i = 0 \qquad l = 1,2,\cdots,m$$

Generalized Coordinates (continue)

The number of degrees of freedom of the system is n = 3N-m. However, when the system is nonholonomic, it is possible to solve the m constraint equations for the corresponding coordinates so that we are forced to work with a number of coordinates exceeding the degrees of freedom of the system. This is permissible provided the surplus number of coordinates matches the number of constraint equations. Although in the case of a holonomic system it may be possible to solve for the excess coordinates, thus eliminating them, this is not always necessary or desirable. If surplus coordinates are used, the corresponding constraint equations must be retained.

The Stationary Value of a Function and of a Definite Integral

In problems of dynamics is often sufficient to find the stationary value of functions instead of the extremum (minimum or maximum).

Definition:
A function is said to have a stationary value at a certain point if the rate of change in every direction of the point is zero.

Examples:

(1) $$f\left(u_1,u_2,\cdots,u_n\right) \quad\rightarrow\quad df=\sum_{i=1}^{n}\frac{\partial f}{\partial u_i}du_i=0 \quad\rightarrow\quad \frac{\partial f}{\partial u_i}=0 \quad i=1,2,\cdots,n$$

By solving those n equations, we obtain $\left(u_1,u_2,\cdots,u_n\right)$ *for which f is stationary*

The Stationary Value of a Function and of a Definite Integral (continue)

Examples (continue):

(2) $f\left(u_1, u_2, \cdots, u_n\right)$ ***with the constraints*** $\sum_{k=1}^{N} a_k^l \, du_k = 0 \qquad l = 1,2,\cdots,m \qquad rank\left\{a_k^l\right\} = m$

Lagrange's multipliers solution gives:

$$df = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial u_i} + \sum_{l=1}^{m} \lambda_l a_i^l\right) du_i = 0$$

By choosing the m Lagrange's multipliers λ_l ***to annihilate the coefficients of the m dependent differentials*** du_i ***we have***

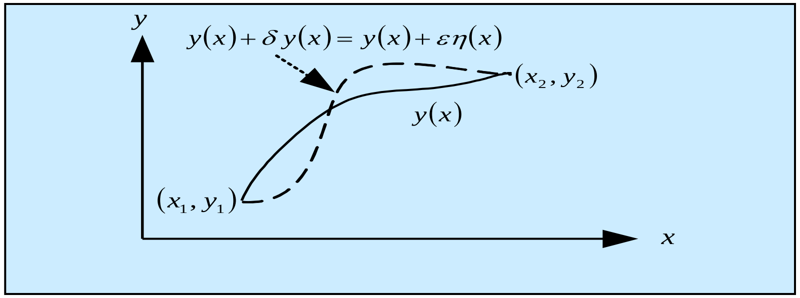
$$\begin{cases} \dfrac{\partial f}{\partial u_i} + \displaystyle\sum_{l=1}^{m} \lambda_l a_i^l = 0 & i = 1,2,\cdots,n \\ \displaystyle\sum_{l=1}^{n} a_i^l du_i = 0 & l = 1,2,\cdots,m \end{cases} \qquad n + m \quad equations$$

The Stationary Value of a Function and of a Definite Integral (continue)

Examples (continue):

(3) ***The functional*** $I = \int_{x_1}^{x_2} F\left(x, y(x), \frac{d\,y(x)}{d\,x} \right) dx$

We want to find $y(x)$ ***such that I is stationary, when the end points*** $y(x_1)$ ***and*** $y(x_2)$ ***are given.***



The variation of $y(x)$ ***is***

$$\bar{y}(x) = y(x) + \delta y(x) = y(x) + \varepsilon \eta(x) \qquad \eta(x_1) = \eta(x_2) = 0$$

and

$$I(\varepsilon) = \int_{x_1}^{x_2} F\left(x, \bar{y}(x), \frac{d\,\bar{y}(x)}{d\,x} \right) dx = \int_{x_1}^{x_2} F\left(x, y(x) + \varepsilon\,\eta(x), \frac{d\,y(x)}{d\,x} + \varepsilon \frac{d\eta(x)}{d\,x} \right) dx$$

$$I(\varepsilon) = I(\varepsilon = 0) + \left.\frac{d\,I}{d\,\varepsilon}\right|_{\varepsilon=0} d\,\varepsilon + \frac{1}{2} \left.\frac{d^2 I}{d\,\varepsilon^2}\right|_{\varepsilon=0} d\,\varepsilon^2 + \cdots$$

The Stationary Value of a Function and of a Definite Integral (continue)

<u>*Examples (continue)*</u>:

Continue: <u>The functional</u> $I = \int_{x_1}^{x_2} F\left(x, y(x), \frac{d\,y(x)}{d\,x}\right) dx$

The necessary condition for a stationary value is

$$\left.\frac{d\,I}{d\,\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial\left(\frac{d\,y}{d\,x}\right)}\frac{d\,\eta(x)}{d\,x}\right)dx$$

$$\underset{by\ \ parts}{\overset{integration}{=}} \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y} - \frac{d}{d\,x}\frac{\partial F}{\partial\left(\frac{d\,y}{d\,x}\right)}\right)\eta(x)dx + \frac{\partial F}{\partial\left(\frac{d\,y}{d\,x}\right)}\underbrace{\left[\eta(x_2) - \eta(x_1)\right]}_{0} = 0$$

JOSEPH-LOUIS LAGRANGE 1736-1813

LEONHARD EULER 1707-1783

Since this must be true for every continuous function $\eta(x)$ ***we have***

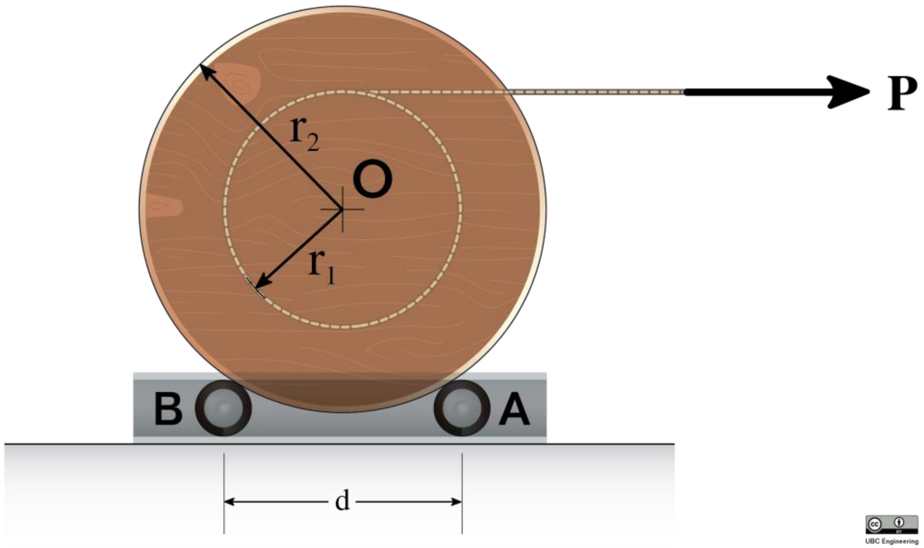
$$\frac{\partial F}{\partial y} - \frac{d}{d\,x}\left[\frac{\partial F}{\partial\left(\frac{d\,y}{d\,x}\right)}\right] = 0 \qquad x_1 \le x \le x_2$$

<u>Euler-Lagrange Differential Equation</u>

By solving this differential equation, $y(x)$ ***,for which I is stationary is found.***

Q1) The turntable on a record player consists of a disk 12 inches in diameter with a weight of 5lbs. The motor accelerates the turntable from rest to its operating speed of 33.33 rpm in one rotation. What is the work done by the motor? What is the average torque the motor exerted?

Q2) A reel of mass 15 kg, resting on two rollers is initially at rest when a force of P = 400 N is applied to a rope attached to the reel. Given that r1 = 0.2 m, r2 = 1 m, and the radius of gyration of the reel is 0.6 m, how many revolutions must the wheel complete to achieve a final angular velocity of 30 rad/s? (Assume no energy is lost due to friction and neglect the mass of the rope and the two rollers)



Q3) A mechanism consists of two, 3 kilogram wheels connected to a 2 kg bar as shown below. Based on the dimensions in the diagram, what is the minimum required initial velocity for the wheels to ensure the mechanism makes it all the way through one rotation without rocking backwards?